Good edge-labelling of graphs.

Julio Araujo, Nathann Cohen, Frédéric Giroire, Frédéric Havet

Good edge-labelling of graphs[☆]

J. Araujo[a,b], N. Cohen[a], F. Giroire[a], F. Havet[a]

[a] *Projet Mascotte I3S (CNRS & Univ. de Nice) and INRIA, INRIA Sophia Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex France.*
[b] *ParGO Research Group, Federal University of Ceará, Campus do Pici, Bloco 910, 60455-760 - Fortaleza - Brazil.*

Abstract

A *good edge-labelling* of a graph G is a labelling of its edges such that, for any ordered pair of vertices (x, y), there do not exist two paths from x to y with increasing labels. This notion was introduced in [2] to solve wavelength assignment problems for specific categories of graphs. In this paper, we aim at characterizing the class of graphs that admit a good edge-labelling. First, we exhibit infinite families of graphs for which no such edge-labelling can be found. We then show that deciding if a graph G admits a good edge-labelling is NP-complete, even if G is bipartite. Finally, we give large classes of graphs admitting a good edge-labelling: C_3-free outerplanar graphs, planar graphs of girth at least 6, $\{C_3, K_{2,3}\}$-free subcubic graphs and $\{C_3, K_{2,3}\}$-free ABC-graphs.

Keywords: Graph Theory, NP-completeness, Edge-labelling, Increasing paths.

1. Introduction

A classical and widely studied problem in WDM (Wavelength Division Multiplexing) networks is the Routing and Wavelength Assignment (RWA) problem [9, 10, 1]. It consists in finding routes, and their associated wavelength as well, to satisfy a set of traffic requests while minimizing the number of used wavelengths. This is a difficult problem which is, in general, NP-hard. Thus, it is often split into two distinct problems: First, routes are found for the requests. Then, in a second step, these routes are taken as an input. Wavelengths must be associated to them in such a way that two routes using the same fiber do not have the same wavelength. The last problem can be reformulated as follows: Given a digraph and a set of dipaths, corresponding to the routes for the requests, find the minimal number of wavelengths w needed to assign different

[☆] Partially supported by the INRIA associated team EWIN between Mascotte and the team ParGO of the Federal University of Ceará, and by the ANR Blanc International Taiwan GRATEL.

Email addresses: `julio.araujo@sophia.inria.fr` (J. Araujo), `nathann.cohen@sophia.inria.fr` (N. Cohen), `frederic.giroire@sophia.inria.fr` (F. Giroire), `frederic.havet@sophia.inria.fr` (F. Havet)

wavelengths to dipaths sharing an edge. This problem can be seen as a colouring problem of the *conflict graph* which is defined as follows: It has one vertex per dipath and two vertices are linked by an edge if their corresponding dipaths share an edge. In [2], Bermond et al. studied the RWA problem for UPP-DAG which are acyclic digraphs (or DAG) in which there is at most one dipath from one vertex to another. In such digraph the routing is forced and thus the unique problem is the wavelength assignment one.

In their paper, they introduce the notion of good edge-labelling. An *edge-labelling* of a graph G is a function $\phi : E(G) \to \mathbb{R}$. A path is *increasing* if the sequence of its edge labels is non-decreasing. An edge-labelling of G is *good* if, for any two distinct vertices u, v, there is at most one increasing (u, v)-path. Bermond et al. [2] showed that the conflict graph of a set of dipaths in a UPP-DAG has a *good edge-labelling*. Conversely, for any graph admitting a good edge-labelling one can exhibit a family of dipaths on a UPP-DAG whose conflict graph is precisely this graph. Bermond et al. [2] then use the existence of graphs with a good edge-labelling and large chromatic number to prove that there exist sets of requests on UPP-DAGs with load 2 (an edge is shared by at most two paths) requiring an arbitrarily large number of wavelengths.

To obtain other results on this problem, it may be useful to identify the *good* graphs which admit a good edge-labelling and the *bad* ones which do not. Bermond et al. [2] noticed that C_3 and $K_{2,3}$ are bad. J.-S. Sereni [12] asked whether every $\{C_3, K_{2,3}\}$*-free graph* (i.e., with no C_3 nor $K_{2,3}$ as a subgraph) is good. In Section 3, we answer this question in the negative. We give an infinite family of bad graphs none of which is the subgraph of another.

Furthermore, in Section 4, we prove that determining if a graph has a good edge-labelling is NP-complete using a reduction from Not-All-Equal 3-SAT.

In Section 5, we show large classes of good graphs: forests, C_3-free outerplanar graphs, planar graphs of girth at least 6. To do so, we use the notion of *critical* graph which is a bad graph such that every proper subgraph of which is good. Clearly, a good edge-labelling of a graph induces a good edge-labelling of all its subgraphs. So every bad graph must contain a critical subgraph. We establish several properties of critical graphs. In particular, we show that they have no *matching-cut*. Hence, a result of Farley and Proskurowski [7, 5] (Theorem 16) implies that a critical graph G has at least $\frac{3}{2}|V(G)| - \frac{3}{2}$ edges.

In Section 6, we use the characterization of graphs with no matching-cut and $\left\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \right\rceil$ edges given by Bonsma [3, 5] to slightly improve this result. We show that a critical graph G has at least $\frac{3}{2}|V(G)| - \frac{1}{2}$ edges unless G is C_3 or $K_{2,3}$.

Finally, we present avenues for future research.

2. Preliminaries

In this section, we give some technically useful propositions. Their proofs are straightforward and left to the reader.

A path is *decreasing* if the sequence of its edge labels is non-increasing. Then, a path $u_1u_2 \dots u_k$ is decreasing if and only if its reversal $u_ku_{k-1} \dots u_1$ is increasing. Hence an edge-labelling is good if and only if for any two distinct vertices u, v, there is at most one decreasing (u,v)-path. Equivalently, an edge-labelling is good if and only if for any two distinct vertices u, v, there is at most one increasing (u,v)-path and at most one decreasing (u,v)-path.

Let x and y be two vertices of G. Two distinct (x,y)-paths P and Q are *independent* if $V(P) \cap V(Q) = \{x,y\}$. Observe that in an edge-labelled graph G, there are two vertices u, v with two increasing (u,v)-paths if and only if there are two vertices u', v' with two increasing independent (u',v')-paths. Hence the definition of good edge-labelling may be expressed in terms of independent paths.

Proposition 1. *An edge-labelling is good if and only if for any two distinct vertices u and v, there are no two increasing independent (u,v)-paths.*

As above the definition may also be in terms of decreasing independent paths. In the paper, we sometimes use Proposition 1 without referring explicitly to it.

Let ϕ be a good edge-labelling of a graph G. If $\phi(E(G)) \subset A$ then for every strictly increasing function $f : A \to B$, $f \circ \phi$ is a good edge-labelling into B. Moreover if ϕ is not injective, one can transform it into an injective one by recursively adding a tiny ϵ to one of the edges having the same label. Hence we have the following.

Proposition 2. *Let G be a graph and A an infinite set in $\mathbb{R} \cup \{-\infty, +\infty\}$. Then G admits a good edge-labelling if and only if it admits an injective good edge-labelling into A.*

Let ϕ be an injective good edge-labelling into an infinite set in $\mathbb{R} \cup \{-\infty, +\infty\}$ of a graph G. Observe that an injective good edge-labelling ϕ' of G into $\mathbb{R}$ can be easily found by just replacing the label $-\infty$ $(+\infty)$ by the smaller (resp., greater) label assigned by ϕ minus (resp., plus) some $\epsilon > 0$.

3. Bad graphs

A path of length one is both increasing and decreasing, and a path of length two is either increasing or decreasing. So C_3 has clearly no good edge-labelling. Also $K_{2,3}$ does not admit a good edge-labelling since there are three paths of length two between the two vertices of degree 3. Hence, in any edge-labelling, two of them are increasing or two of them are decreasing.

Extending this idea, we now construct an infinite family of bad graphs, none of which is the subgraph of another. The construction of this family is based on the graphs H_k defined below. These graphs play the same role as a path of length two because they have two vertices u and v such that any good edge-labelling of H_k has either a (u,v)-increasing path or a (v,u)-increasing path.

For any integer $k \geq 3$, let H_k be the graph defined by

$$\begin{aligned} V(H_k) &= \{u,v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_i \mid 1 \leq i \leq k\}, \\ E(H_k) &= \{uu_i \mid 1 \leq i \leq k\} \cup \{u_iv_i \mid 1 \leq i \leq k\} \cup \{v_iv \mid 1 \leq i \leq k\}, \\ &\quad \cup \{v_iu_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

with $u_{k+1} = u_1$. See Figure 1.

Observe that the graph H_k has no $K_{2,3}$ as a subgraph, and for $i \neq k$, H_i is not a subgraph of H_k.

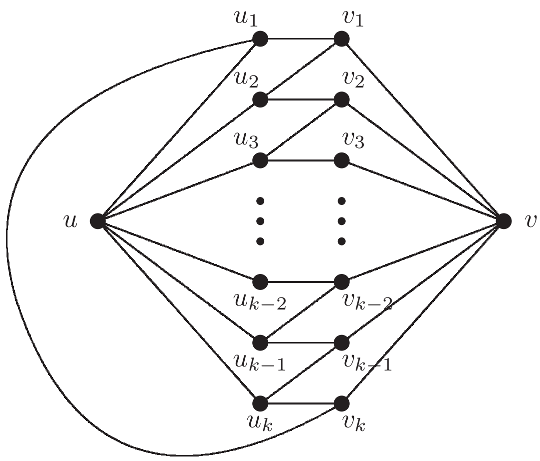

Figure 1: Graph H_k

Proposition 3. *Let $k \geq 3$. For every good edge-labelling, the graph H_k has either an increasing (u, v)-path or an increasing (v, u)-path.*

Proof: Suppose, by way of contradiction, that H_k has a good edge-labelling ϕ having no increasing (u, v)-path and no increasing (v, u)-path. By Proposition 2, we may assume that ϕ is injective.

A key component in this proof is the following observation which follows easily from the fact that ϕ is good.

Observation 3.1. *Suppose $x_1x_2x_3x_4x_1$ is a 4-cycle. Then, either*

- $\phi(x_4x_1) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_3x_4)$ *and* $\phi(x_1x_4) < \phi(x_3x_4)$*; or*
- *all those inequalities are reversed.*

By symmetry, we may assume that $\phi(uu_1) < \phi(u_1v_1)$. By Observation 3.1, $\phi(v_1u_2) < \phi(u_1v_1)$, $\phi(v_1u_2) < \phi(uu_2)$ and $\phi(uu_1) < \phi(uu_2)$. Then, since vv_1u_2u is not increasing, $\phi(u_2v_1) < \phi(v_1v)$. Again by Observation 3.1, $\phi(v_2v) < \phi(u_2v_2)$. Thus since uu_2v_2v is not increasing $\phi(uu_2) < \phi(u_2v_2)$.

Applying the same reasoning, we obtain that $\phi(uu_2) < \phi(uu_3)$ and $\phi(uu_3) < \phi(u_3v_3)$ and so on, iteratively, $\phi(uu_1) < \phi(uu_2) < \cdots < \phi(uu_k) < \phi(uu_1)$, a contradiction. $\square$

For convenience we denote by H_2 the path of length 2 with end vertices u and v. Let i, j, k be three integers greater than 1. The graph $J_{i,j,k}$ is the graph obtained from disjoint copies of H_i, H_j and H_k by identifying the vertices u of the three copies and the vertices v of the three copies.

Proposition 4. *Let i, j, k be three integers greater than 1. Then $J_{i,j,k}$ is bad.*

Proof: Suppose, by way of contradiction, that $J_{i,j,k}$ admits a good edge-labelling. By Proposition 3, in each of the subgraphs H_i, H_j and H_k, there is either an increasing (u, v)-path or an increasing (v, u)-path. Hence in $J_{i,j,k}$, there are either two increasing (u, v)-paths or two increasing (v, u)-paths, a contradiction. □

4. *NP*-completeness

In this section, we prove that it is an NP-complete problem to decide if a bipartite graph admits a good edge-labelling. We give a reduction from the NOT-ALL-EQUAL (NAE) 3-SAT Problem [11] which is defined as follows:

Instance: A set V of variables and a collection $\mathcal{C}$ of clauses over V such that each clause has exactly 3 literals.

Question: Is there a truth assignment such that each clause has at least one true and at least one false literal?

For sake of clarity, we first present the NP-completeness proof for general graphs.

Theorem 5. *The following problem is NP-complete.*
Instance: *A graph G.*
Question: *Does G have a good edge-labelling?*

Proof: Given a graph G and an injective edge-labelling ϕ into $\mathbb{R}$, one can check in polynomial time if ϕ is good or not using the following algorithm where $(u_1v_1, \ldots, u_mv_m)$ is an ordering of the edges of G in increasing order according to their labels.

```
foreach u ∈ V(G) do
    Set V(T) := {u}, E(T) := ∅;
    foreach i=1 to m do
        if {u_i, v_i} ⊂ V(T) then
            return "bad edge-labelling";
        if u_i ∈ V(T) (and v_i ∉ V(T)) then
            V(T) := V(T) ∪ {v_i} and E(T) := E(T) ∪ {u_i v_i};
return "good edge-labelling";
```

Indeed, for each vertex u, the above algorithm grows the tree T of increasing paths from u: at each step i, T is the tree of increasing paths from u with arcs with labels less than $\phi(u_iv_i)$. In particular, there is an increasing (u, v)-path P_v

for every $v \in V(T)$. Hence if $u_i \in V(T)$ and $v_i \in V(T)$ then P_{v_i} and $P_{u_i} + u_i v_i$ are two increasing (u, v_i)-paths, so the edge-labelling is not good. If $u_i \in V(T)$ and $v_i \notin V(T)$, then $P_{u_i} + u_i v_i$ is a new increasing path that must be included into T. Finally, if $u_i \notin V(T)$ and $v_i \notin V(T)$, then $u_i v_i$ will not be in any increasing path from u as the edges to be considered after it have larger labels.

Hence the considered problem is in NP.

To prove that the problem is NP-complete, we will reduce the NAE 3-SAT Problem without repetition (i.e. a variable appears at most once in each clause) which is equivalent to NAE 3-SAT Problem (with repetition) to it. (For each repeated variable x, we introduce two other variables y and z. Then the second (third) occurrence of x in a clause is replaced by y (z). Then, x, y, z are forced to have the same truth assignment by adding $\bar{x} \vee y \vee z$, $x \vee \bar{y} \vee z$, $x \vee y \vee \bar{z}$, $\bar{x} \vee \bar{y} \vee z$, $\bar{x} \vee y \vee \bar{z}$, and $x \vee \bar{y} \vee \bar{z}$ to the instance.)

Let $V = \{x_1, \ldots, x_n\}$ and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be an instance I of the NAE 3-SAT Problem without repetition. We shall construct a graph G_I in such a way that I has an answer yes for the NAE 3-SAT Problem if and only if G_I has a good edge-labelling.

For each variable x_i, $1 \le i \le n$, we create a variable graph VG_i defined as follows (See Figure 2.):

$$\begin{aligned}
V(VG_i) &= \{v_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \cup \{r_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \\
&\quad \cup \{s_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\}. \\
E(VG_i) &= \{v_k^{i,j} v_{k+1}^{i,j} \mid 1 \le j \le m, 1 \le k \le 3\} \cup \{v_4^{i,j} v_1^{i,j+1} \mid 1 \le j \le m-1\} \\
&\quad \cup \{v_k^{i,j} r_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \cup \{v_k^{i,j} s_k^{i,j} \mid 1 \le j \le m, 1 \le k \le 4\} \\
&\quad \cup \{v_4^{i,j} r_1^{i,j} \mid 1 \le j \le m\} \cup \{v_k^{i,j+1} r_{k+1}^{i,j} \mid 1 \le j \le m-1, 1 \le k \le 3\} \\
&\quad \cup \{v_4^{i,j} s_1^{i,j} \mid 1 \le j \le m\} \cup \{v_k^{i,j+1} s_{k+1}^{i,j} \mid 1 \le j \le m-1, 1 \le k \le 3\}.
\end{aligned}$$

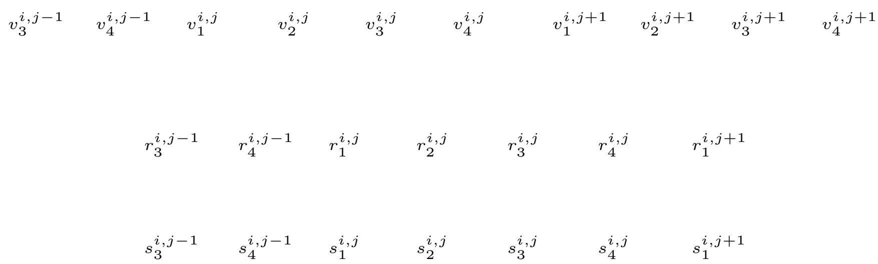

Figure 2: The variable graph VG_i

For each clause $C_j = l_1 \vee l_2 \vee l_3$, $1 \le j \le m$, we create a clause graph CG_j defined as follows (See Figure 3.):

$$\begin{aligned}
V(CG_j) &= \{c^j, b_1^j, b_2^j, b_3^j\}; \\
E(CG_j) &= \{c^j b_1^j, c^j b_2^j, c^j b_3^j\}.
\end{aligned}$$

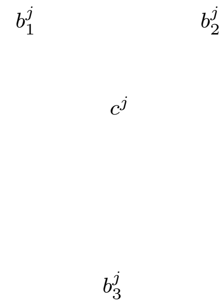

Figure 3: The clause graph CG_j.

Now, for each literal l_k, $1 \leq k \leq 3$, if l_k is the non-negated variable x_i, we identify b_k^j, c^j and b_{k+1}^j (index k is taken modulo 3) with $v_1^{i,j}$, $v_2^{i,j}$ and $v_3^{i,j}$, respectively. Otherwise, if l_k is the negated variable $\bar{x_i}$, we identify b_k^j, c^j and b_{k+1}^j with $v_3^{i,j}$, $v_2^{i,j}$ and $v_1^{i,j}$, respectively.

Let us first show that, if G_I has a good edge-labelling ϕ, then there is a truth assignment such that each clause of I has at least one true literal and at least one false literal.

By Proposition 2, we may assume that ϕ is injective.

Claim 5.1. *Let* $1 \leq i \leq n$. *If* $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ *then* $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ *for all* $1 \leq j \leq m$.

Proof: By induction on j. A path of length two is necessarily increasing or decreasing. Now $v_1^{i,j}$ is joined to $v_4^{i,j}$ by two paths of length two via $r_1^{i,j}$ and $s_1^{i,j}$. Since ϕ is good, one of these two paths is increasing and the other one is decreasing. In addition, the path $v_1^{i,j}v_2^{i,j}v_3^{i,j}v_4^{i,j}$ is neither increasing nor decreasing so $\phi(v_2^{i,j}v_3^{i,j}) > \phi(v_3^{i,j}v_4^{i,j})$.

Applying three times this reasoning, we derive $\phi(v_3^{i,j}v_4^{i,j}) < \phi(v_4^{i,j}v_1^{i,j+1})$, $\phi(v_4^{i,j}v_1^{i,j+1}) > \phi(v_1^{i,j+1}v_2^{i,j+1})$ and finally $\phi(v_1^{i,j+1}v_2^{i,j+1}) < \phi(v_2^{i,j+1}v_3^{i,j+1})$. $\square$

Hence we define the truth assignment Λ by $\Lambda(x_i) = \mathit{true}$ if $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ and $\Lambda(x_i) = \mathit{false}$ otherwise.

Let us show that each clause C_j has at least one true literal or one false literal. Set $C_j = l_1 \vee l_2 \vee l_3$. First observe that, by construction, for all $1 \leq k \leq 3$, l_k is true if $\phi(b_k^j c^j) < \phi(b_{k+1}^j c^j)$ and l_k is false otherwise. Now the three literals are not all true otherwise, $\phi(b_1^j c^j) < \phi(b_2^j c^j) < \phi(b_3^j c^j) < \phi(b_1^j c^j)$, a contradiction. And they are not all false, otherwise $\phi(b_1^j c^j) > \phi(b_2^j c^j) > \phi(b_3^j c^j) > \phi(b_1^j c^j)$, a contradiction. Hence C_j has at least one true literal and one false literal.

Conversely, let us now show that if there is a truth assignment Λ such that each clause of I has at least one true literal and at least one false literal, then G_I has a good edge-labelling.

The idea is to find a good edge-labelling ϕ satisfying the following property ($\star$): If $\Lambda(x_i) = true$, $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$ and if $\Lambda(x_i) = false$, $\phi(v_1^{i,j}v_2^{i,j}) > \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$.

Let $C_j = l_1 \vee l_2 \vee l_3$ be clause. To satisfy ($\star$), we must label the edges of VG_j such that $\phi(b_k^j c^j) < \phi(b_{k+1}^j c^j)$ if l_k is true and $\phi(b_k^j c^j) > \phi(b_{k+1}^j c^j)$ if l_k is false. As C_j has at least one true and one false literal, there is a unique way to label the three edges $c^j b_k^j$, $1 \leq k \leq 3$, with $\{-1, 0, +1\}$ such that the condition ($\star$) is fulfilled.

Let us now extend this edge-labelling to the remaining edges of each of the variable graphs VG_i. First, for all $1 \leq j \leq m$ and $1 \leq k \leq 4$, assign -3 and $+3$ alternatingly on the edges of the cycle of length four containing both $r_k^{i,j}$ and $s_k^{i,j}$ such that $\phi(v_k^{i,j}r_k^{i,j}) = -3$. Then, if $\Lambda(x_i) = true$, set $\phi(v_3^{i,j}, v_4^{i,j}) = -2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = 2$ for all $1 \leq j \leq m$, and, if $\Lambda(x_i) = false$, set $\phi(v_3^{i,j}, v_4^{i,j}) = 2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = -2$ for all $1 \leq j \leq m$.

We claim that ϕ is a good edge-labelling of G_I. Suppose, by way of contradiction, that there is a pair of vertices (x, y) such that two independent increasing (x, y)-paths P_1 and P_2 exist.

A set of two independent paths starting at a vertex of $R = \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}$ contains one increasing path (the one starting with the edge labelled -3) and one decreasing path (the one starting with the edge labelled 3). Hence x and y are not in R.

In addition, the union of P_1 and P_2 cannot be one of the four-cycles formed by the edges incident to $r_k^{i,j}$ and $s_k^{i,j}$ for some i,j and k.

Without loss of generality, we may assume that P_1 is at least as long as P_2. As cycles formed by two graphs GV_i and GV_j are of length at least 6, P_1 has length at least 3. Now one can see that P_1 may not contain any vertex of R because every path of length at least 3 with internal vertices in R is not increasing (nor decreasing).

Hence P_1 must contain at least three consecutive edges on one of the paths $Q_i = VG_i - R$. So P_1 is not increasing, a contradiction. □

Observe that the graph G_I constructed in the above proof is not bipartite. However, with a slight modification, we can transform it into a bipartite graph and obtain the following theorem.

Theorem 6. *The following problem is NP-complete.*
Instance: *A bipartite graph G.*
Question: *Does G have a good edge-labelling?*

Proof: Let G'_I be the graph obtained from G_I (described in the proof of Theorem 5) by replacing each path $v_k^{i,j}, r_k^{i,j}, v_{k+3}^{i,j}$ and each path $v_k^{i,j}, s_k^{i,j}, v_{k+3}^{i,j}$, by copies of a graph $H_{k'}$ defined in Section 4, for some $k' \geq 3$ and for all $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k = 1, \ldots, 4$ ($k + 3$ is taken modulo 4).

By Proposition 3, it is not difficult to verify that G'_I admits a good edge-labelling if, and only if, G'_I also does. Moreover, each $H_{k'}$ admits a proper 2-colouring such that the vertices u and v have disjoint colours. Thus, G'_I is bipartite, since it admits a proper 2-colouring where all the vertices $v_1^{i,j}$ and $v_3^{i,j}$ belong to the same colour class, for all $i = 1, \dots, n$ and $j = 1, \dots, m$. □

5. Classes of good graphs

Recall that a graph G is *critical* if it is bad but each of its proper subgraphs is good. To prove that every graph in a class $\mathcal{C}$ closed under taking subgraphs has a good edge-labelling, it suffices to prove that $\mathcal{C}$ contains no critical graph.

Lemma 7. *Let G be a graph with a cutvertex x, $C_1, \dots, C_k$ be the components of $G - x$ and $G_i = G\langle C_i \cup \{x\}\rangle$, $1 \le i \le k$. Then G is good if and only if all the G_i are good.*

Proof: Necessity is obvious since a good edge-labelling of G induces a good edge-labelling on each subgraph G_i.

Sufficiency follows from the fact that there are two independent (u, v)-paths in G only if there exists i, $1 \le i \le k$, such that u and v are in $V(G_i)$. Hence the union of good edge-labellings of all the G_i is a good edge-labelling of G. □

Corollary 8. *Every critical graph is 2-connected.*

Corollary 9. *Every forest F admits a good edge-labelling.*

Proof: No forest contains a non-trivial 2-connected subgraph, and so contains no critical subgraph. □

Let $G = (V, E)$ be a graph. A *K_2-cut* of G is a set of two adjacent vertices u and v such that the graph $G - \{u, v\}$ (obtained from G by removing u and v and their incident edges) has more connected components than G.

Lemma 10. *Let G be a connected graph and $\{u, v\}$ a K_2-cut in G such that $G - \{u, v\}$ has two connected components C_1 and C_2. If $G_1 = G\langle C_1 \cup \{u, v\}\rangle$ and $G_2 = G\langle C_2 \cup \{u, v\}\rangle$ have a good edge-labelling then G has a good edge-labelling.*

Proof: Let ϕ_1 and ϕ_2 be good edge-labellings of $G\langle C_1 \cup \{u, v\}\rangle$ and $G\langle C_2 \cup \{u, v\}\rangle$ respectively such that $\phi_1(uv) = \phi_2(uv)$.

Then the union of ϕ_1 and ϕ_2 is a good edge-labelling of G. Indeed, suppose by way of contradiction, that there exists x and y and two independent increasing (x, y)-paths P_1 and P_2 in G. W. l. o. g., we may assume that $x \in V(G_1)$. At least one of the paths, say P_1, contains at least one edge e_1 in $E(G_2) \setminus \{uv\}$ since ϕ_1 is good.

Assume first that $y \in V(G_1)$. Then P_1 must go through u and v. Let Q_2 be the shortest (u, v)-subpath of P_1 containing e_1. Then Q_2 is either increasing or decreasing. Hence since uv is both increasing and decreasing, there are either two increasing paths or two decreasing paths in G_2. This contradicts the fact that ϕ_2 is good.

Assume now that $y \in C_2$. Then since P_1 and P_2 are independent without loss of generality, P_1 goes through u and P_2 goes through v. Let Q_1 be the (x, u)-subpath of P_1, R_1 be the (u, y)-subpath of P_1, let Q_2 be the (x, v)-subpath of P_2 and R_2 be the (v, y)-subpath of P_2.

If $\phi(uv)$ is larger than the label of the last edge of Q_1, then Q_1uv and Q_2 are two increasing (x, v)-paths in G_1, a contradiction. If not $\phi(uv)$ is smaller than the label of the first edge of R_1 and vuR_1 and R_2 are two increasing (v, y)-paths in G_2, a contradiction.

□

Let $G = (V, E)$ be a graph. An *edge-cut* is a non-empty set of edges between a set of vertices S and its complement $\overline{S}$. Formally, for any $S \subset V$, the edge-cut $[S, \overline{S}]$ is the set $\{uv \in E \mid u \in S \text{ and } v \in \overline{S})$. An edge cut which is also a matching is called a *matching-cut*.

Lemma 11. *Let G be a graph and $[S, \overline{S}]$ a matching-cut in G. If $G\langle S\rangle$ and $G\langle\overline{S}\rangle$ have a good edge-labelling then G has a good edge-labelling.*

Proof: Let ϕ_1 be a good edge-labelling of $G\langle S\rangle$ and ϕ_2 be a good edge-labelling of $G\langle\overline{S}\rangle$ (in $\mathbb{R}$). Then the edge-labelling ϕ of G defined by $\phi(e) = \phi_1(e)$ if $e \in E(G\langle S\rangle)$, $\phi(e) = \phi_2(e)$ if $e \in E(G\langle\overline{S}\rangle)$ and $\phi(e) = +\infty$ if $e \in [S, \overline{S}]$ is good.

Indeed, suppose by way of contradiction, that it is not good. Then there exist two vertices u and v and two independent increasing (u, v)-paths P_1 and P_2. Since ϕ_1 and ϕ_2 are good, then without loss of generality, we may assume that $u \in S$ and $v \in \overline{S}$. For $i = 1, 2$, the path P_i contains an edge of u_iv_i in $[S, \overline{S}]$. Now, since u_1v_1 and u_2v_2, are labelled $+\infty$ and incident to no edges labelled $+\infty$, u_1v_1 must be the last edge of P_1 and u_2v_2 the last edge of P_2. So $v_1 = v = v_2$, which is impossible as $[S, \overline{S}]$ is a matching. □

Corollary 12. *A critical graph has no matching-cut.*

Corollary 13. *Every C_3-free outerplanar graph admits a good edge-labelling.*

Proof: An easy result of Eaton and Hull [6] states that a C_3-free outerplanar graph has either a vertex of degree 1 or two adjacent vertices of degree 2. This implies that it has a matching-cut. Hence by Corollary 12 no C_3-free outerplanar graph is critical, which yields the result. □

A graph is *subcubic* if every vertex has degree at most three.

Lemma 14. *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a matching-cut.*

Proof: Let G be a subcubic graph $\{C_3, K_{2,3}\}$-free. If G has no cycle, then every edge forms a matching-cut. Suppose now that G has a cycle. Let C be a cycle of smallest length in G. If C is a connected component of G (in particular if $C = G$) then any pair of non-adjacent edges of C forms a matching-cut.

If not, let us show that $[V(C), \overline{V(C)}]$ is a matching-cut. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two distinct edges in $[V(C), \overline{V(C)}]$ with $x_1, x_2 \in V(C)$. Then $x_1 \neq x_2$ because these two vertices have degree (at most) 3 and they have two neighbours in $V(C)$. Suppose by way of contradiction that $y_1 = y_2$. Then x_1 and x_2 are not adjacent since G is C_3-free. Furthermore, there are the two (x_1, x_2)-paths along C are of length at most 2 otherwise C would not be a smallest cycle. Hence C is a cycle of length 4 and the graph induced by $V(C) \cup \{y_1\}$ is a $K_{2,3}$, a contradiction. □

Corollary 12 and Lemma 14 immediately imply that the sole subcubic critical graphs are C_3 and $K_{2,3}$.

Corollary 15. *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a good edge-labelling.*

Farley and Proskurowski [7, 5] proved that every (multi)graph G on n vertices with less than $\frac{3}{2}(n-1)$ edges has a matching-cut.

Theorem 16 (Farley and Proskurowski [7, 5]). *Let G be a multigraph. If $|E(G)| < \frac{3}{2}|V(G)| - \frac{3}{2}$ then G has a matching-cut.*

Corollary 12 and Theorem 16 yield immediately the following.

Corollary 17. *Every critical graph has at least $\left\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \right\rceil$ edges.*

An easy and well-known consequence of Euler's Formula states that every planar graph with girth at least 6 has at most $\frac{3}{2}|V(G)| - 3$ edges and so is not critical.

Corollary 18. *Every planar graph of girth at least 6 has a good edge-labelling.*

6. Good edge-labelling of ABC-graphs

Corollary 17 states that every critical graph has at least $\left\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \right\rceil$ edges. This is tight since if G is C_3 or $K_{2,3}$ then $|E(G)| = \left\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \right\rceil$. We will now show that those two graphs are the unique critical ones satisfying this equality.

Farley and Proskurowski [7, 5] constructed a class of multigraphs G (called *ABC-graphs*) having $\left\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \right\rceil$ edges with no matching-cut. The definition of ABC-graphs is based on the following three operations:

- An *A-operation* on vertex u introduces vertices v and w and edges uv, uw and vw.
- A *B-operation* on edge uv introduces vertices w_1 and w_2 and edges uw_1, vw_1, uw_2 and vw_2, and removes edge uv.

- A *C-operation* on vertices u and v ($u = v$ is allowed) introduces vertex w and edges uw and vw.

Note that the C-operation is the only operation that can introduce parallel edges.

An *ABC-graph* is a graph that can be obtained from K_1 with a sequence of A- and B-operations and at most one C-operation.

It is easy to check that ABC-graphs have no matching-cut. In addition, solving a conjecture of Farley and Proskurowski, Bonsma [3, 5] showed that they are the unique extremal examples, i.e., satisfying $|E(G)| = \lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$.

Theorem 19 (Bonsma [3, 5]). *Let G be a graph such that $|E(G)| = \lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$. Then G has no matching-cut if and only if G is an ABC-graph.*

Our aim is to prove that every $\{C_3, K_{2,3}\}$-free ABC-graph is good. It is easy to check that every 2-connected component of an ABC-graph is an ABC-graph, so by Lemma 7, it suffices to prove it for 2-connected ABC-graphs.

Observe that the C-operation is the only one that changes the parity of the order. Hence an ABC-graph with an odd number of vertices is obtained from K_1 with a sequence of A- and B-operations and no C-operation.

Let G be a graph obtained from a graph H by a B-operation on some edge uv. Let ϕ be an edge-labelling of H. Let ϕ_0 and ϕ_∞ be the edge-labellings of G defined by:

$$\begin{aligned}
&\phi_0(e) = \phi_\infty(e) = \phi(e) \text{ for all } e \in E(H) \setminus \{uv\},\\
&\phi_0(uw_1) = \phi_0(w_2v) = 1/2,\\
&\phi_0(uw_2) = \phi_0(w_1v) = -1/2,\\
&\phi_\infty(uw_1) = \phi_\infty(w_2v) = +\infty,\\
&\phi_\infty(uw_2) = \phi_\infty(w_1v) = -\infty
\end{aligned}$$

Proposition 20. *Let G be a graph obtained from a graph H by a B-operation on some edge uv and ϕ be a good edge-labelling of H.*

(i) If ϕ is injective integer-valued and $\phi(uv) = 0$, then ϕ_0 is a good edge-labelling of G.

(ii) If ϕ is real-valued, then ϕ_∞ is a good edge-labelling of G.

Proof: (i) By contradiction, suppose that ϕ_0 is not a good edge-labelling of G. Then there exist two increasing independent (x, y)-paths P_1 and P_2 on G, for some $x, y \in V(G)$.

Since ϕ is a good edge-labelling of H, by the definition of ϕ_0 at least one edge of the set $E' = \{uw_1, uw_2, vw_1, vw_2\}$ belongs to some of the paths P_1 or P_2. Observe also that an increasing path in H cannot contain more than two edges of E'.

Suppose then that exactly one of the paths, say P_1, contains a non-empty intersection with the set E'. In this case, there would be two increasing paths in the edge-labelling ϕ of H. To prove this fact, let P_1' be the path obtained from P_1 by replacing the edges of the set $E' \cap E(P_1)$ by the edge uv. Observe that P_1' and P_2 would be two increasing paths of H under the edge-labelling ϕ, since $\phi(uv) = 0$.

Hence the paths P_1 and P_2 both contain some edge of the set E'. Suppose first that P_1 and P_2 contain exactly one edge of E' each. As P_1 and P_2 are independent, we assume that $uw_1 \in E(P_1)$ and $vw_1 \in E(P_2)$, without loss of generality. If $w_1 = y$, then the last edge of the (x, u)-subpath of P_1 has a label smaller than 0 (since ϕ is injective) and the same happens for the last edge of the (x, v)-subpath of P_2 (observe that at least one of these subpaths must be non-empty). Consequently, there would be two increasing paths (x, u)-paths or (x, v)-paths in H under the edge-labelling ϕ. Similarly, one may conclude that if $w_1 = x$, then there would also be two increasing paths on ϕ. It is just necessary to verify that the first edges of the (u, y)-subpath of P_1 and of the (v, y)-subpath of P_2 are greater than 0 (at least one of these edges exist) and that there would be two increasing (u, y)-paths or (v, y)-paths in H.

Finally, P_1 and P_2 cannot have both two edges from E' because they are independent.

(ii) The proof that ϕ_∞ is a good edge-labelling of G is similar to the proof of (i). In this case, P_1 and P_2 cannot contain just one edge of E'. Consequently, either $E(P_1) \subset E'$ or $E(P_2) \subset E'$. In any case, there would be and increasing (u, v)-path or an increasing (v, u)-path, which is a contradiction because there would be two increasing paths in H. □

Corollary 21. *If G is a graph obtained from a good graph by a B-operation, then G is good.*

Proof: It follows directly from Proposition 20. □

Lemma 22. *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G is good.*

Proof: By contradiction, suppose that G is a counter-example to the statement. As every A-operation (with the exception of the transition $K_1 \to C_3$) creates a cut-vertex, by Lemma 7, we may assume that G is obtained from C_3 with a sequence of B-operations. However a B-operation on C_3 at any edge creates a $K_{2,3}$ and a B-operation on $K_{2,3}$ at any edge creates the graph G_1 depicted in Figure 4. If $G \notin \{C_3, K_{2,3}\}$ then it is obtained from G_1 with a sequence of B-operations. Now this graph G_1 admits a good edge-labelling (See Figure 4). Hence an easy induction and Corollary 21 imply that G has a good edge-labelling, a contradiction.

□

Since 2-connected components of an ABC-graph with an odd number of vertices are ABC-graphs with an odd number of vertices, we have the following:

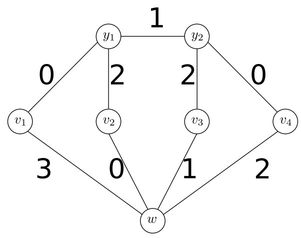

Figure 4: The graph G_1 and a good edge-labelling.

Corollary 23. *Every $\{C_3, K_{2,3}\}$-free ABC-graph with an odd number of vertices is good.*

We now would like to prove an analogous statement to the one of Corollary 23 but for ABC-graphs with an even number of vertices.

Let G be a graph and x, y be two distinct vertices of G. An (x, y)-*better edge-labelling* of G is a good edge-labelling of G such that there is no increasing (x, y)-path. Clearly, if x and y are adjacent or if x and y have two neighbours in common then G has no (x, y)-better edge-labelling. A graph is *friendly* if it has a good edge-labelling and for any pair (x, y) of non-adjacent vertices with at most one neighbour in common there exists an (x, y)-better edge-labelling.

Let G_1 be a graph whose vertex set is $\{v_1, v_2, v_3, v_4, w, y_1, y_2\}$ and whose edge set is $\bigcup_{i=1}^{4}\{(w, v_i)\} \cup \{(v_1, y_1), (v_2, y_1), (v_3, y_2), (v_4, y_2)\} \cup \{y_1, y_2\}$ (See Figure 4.).

Lemma 24. *G_1 is friendly.*

Proof: Let ϕ be the edge-labelling of G_1 in Figure 4. Then ϕ is good.

Let us now prove that for every pair $p = (a, b)$ of two distinct non-adjacent vertices a and b in G_1 such that a and b have at most one common neighbour, there is a better (a, b)-edge-labelling of G_1.

First, observe that the vertex w of G_1 cannot be in such a pair because, for any other vertex of G_1, either w is adjacent to it or they have two common neighbours.

Suppose now that the vertex $y_1 \in p$. Then the other vertex of p must be v_3 or v_4. But ϕ is (v_3, y_1)-better and (y_1, v_4)-better, and so $-\phi$ is (y_1, v_3)-better and (v_4, y_1)-better. Hence in any case, there is a better p-edge-labelling of G_1.

By symmetry, if y_2 is a vertex of p, there exists a p-better edge-labelling.

Suppose that $v_1 \in p$. Then the other vertex of p is v_3 or v_4. ϕ is (v_1, v_4)-better and exchanging the labels of y_2v_3 and y_2v_4 and also the labels of v_3w and v_4w we obtain a (v_1, v_3)-better edge-labelling ϕ'. Thus $-\phi'$ and $-\phi$ are respectively (v_3, v_1)-better and (v_4, v_1)-better. Hence in any case, there is a better p-edge-labelling of G_1.

By symmetry, if v_2, v_3 or v_4 is a vertex of p, there exists a p-better edge-labelling. $\square$

Proposition 25. *Let G be a graph obtained from a graph H by a B-operation on some edge uv. If H is friendly then G is friendly.*

Proof: Let w_1, w_2 be the vertices created by the B-operation. Let x and y be two non-adjacent vertices of G having at most one neighbour in common. Then $|\{x, y\} \cap \{w_1, w_2\}| \leq 1$.

- Suppose first that $\{x, y\} \cap \{w_1, w_2\} = \emptyset$. Then x and y are not adjacent in H.

  Assume first that x and y have at most one common neighbour in H. Let ϕ be an injective integer-valued (x, y)-better edge-labelling of H such that $\phi(uv) = 0$. Then ϕ_0 is a good edge-labelling of G by Proposition 20-(i). Moreover it is easy to check that there is no increasing (x, y)-path in G. Hence ϕ_0 is an (x, y)-better edge-labelling of G.

  Assume now that x and y have two common neighbours in H. As they do not have two common neighbours in G, we can suppose w.l.o.g. that $x = u$ and $N(x) \cap N(y) = \{v, w\}$, for some vertex w. Let ϕ be a real-valued good edge-labelling of H. Free to consider $-\phi$, we may assume that uvy is an increasing path. Hence in $H \setminus uv$ there is no increasing (u, y)-path. By Proposition 20-(ii), ϕ_∞ is a good edge-labelling of G. Moreover it is an (x, y)-better edge-labelling, because there is no increasing (u, y)-path in $H \setminus uv$ and the unique increasing paths containing w_1 and w_2 are uw_2 and uw_2v.

- Suppose now that $|\{x, y\} \cap \{w_1, w_2\}| = 1$. Without loss of generality, we may assume that $x = w_1$ and y is not adjacent to v.

  Assume first that v and y have at most one common neighbour in H. Let ϕ be a (v, y)-better edge-labelling of H. By Proposition 2, we may assume that ϕ is real-valued. By Proposition 20-(ii), ϕ_∞ is a good edge-labelling of G. Moreover, there is no increasing (w_1, y)-path, through u since $\phi(uw_1) = +\infty$, nor through v since there is no increasing (v, y)-path in H. Hence ϕ_∞ is a (w_1, y)-better edge-labelling of G.

  Assume now that v and y have two common neighbours in H.

  - Suppose that y is adjacent to u. Let ϕ be an injective integer-valued good edge-labelling of H such that $\phi(uv) = 0$. Free to consider $-\phi$, we may assume that $\phi(uy) < 0$ and so $\phi(uy) \leq -1$. By Proposition 20-(i), ϕ_0 is a good edge-labelling of G. Moreover it has no increasing (w_1, y)-path and so is (w_1, y)-better. Indeed suppose for a contradiction that there is an increasing (w_1, y)-path P :

    * If u is the second vertex of P then $P - w_1$ is an increasing (u, y)-path. Since $\phi(uy) \leq -1$, $P - w_1$ is not (u, y). So $P - w_1$ and (u, y) are two increasing (u, y)-paths in H a contradiction.
    * If v is the second vertex of P then the path Q in H obtained from P by replacing w_1 with u is an increasing (u, y)-path because the

labels of the edges of $P - w_1$ are positive. Thus Q and (u, y) are distinct increasing (u, y)-paths, a contradiction.

– Suppose that y is not adjacent to u. Let t_1 and t_2 be the two common neighbours of v and y. Let ϕ be an injective integer-valued good edge-labelling of H such that $\phi(uv) = 0$. Without loss of generality, we may assume that (v, t_1, y) is increasing and (v, t_2, y) is decreasing. By Observation 3.1, $\phi(vt_1) < \phi(vt_2)$. Thus, if $\phi(vt_1) > 0$ then $\phi(vt_2) > 0$. So with respect to $-\phi$, (v, t_2, y) is increasing and $-\phi(vt_2) < 0$. Hence, free to consider $-\phi$ (and swap the names of t_1 and t_2), we may assume that $\phi(vt_1) < 0$ and so $\phi(vt_1) \leq -1$. By Proposition 20-(i), ϕ_0 is a good edge-labelling of G. Moreover it has no increasing (w_1, y)-path and so is (w_1, y)-better. Indeed suppose for a contradiction that there is a increasing (w_1, y)-path P :

  * If v is the second vertex of P then $P - w_1$ is an increasing (v, y). Since $\phi(vt_1) \leq -1$, $P - w_1$ is not (v, t_1, y). So there are two increasing (v, y)-paths in H, a contradiction.
  * If u is the second vertex of P then the path P' in H obtained from P by replacing w_1 with v is an increasing (v, y)-path because the labels of the edges of $P - w_1$ are positive. P' is distinct from (v, t_1, y), a contradiction.

□

One can now generalize Lemma 22.

Lemma 26. *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G is friendly.*

Proof: Similarly as in the proof of Lemma 22, combining Lemma 24 and Proposition 25 yield the result by induction. □

Corollary 27. *Every $\{C_3, K_{2,3}\}$-free ABC-graph with an odd number of vertices is friendly.*

Proof: Let x and y be two non-adjacent vertices of G having at most one common neighbour.

Assume first that x and y are in a same connected 2-component C. By Lemma 26, C has an (x, y)-better edge-labelling and, by Corollary 23, $G \setminus E(C)$ has a good edge-labelling. The union of these two edge-labellings is clearly an (x, y)-better labelling of G.

Suppose now that the 2-connected components containing x do not contain y. Let G_1 be the graph induced by the union of the 2-connected components containing x and $G_2 = G \setminus E(G_1)$. By Corollary 23, the two graphs G_1 and G_2 admit good edge-labellings ϕ_1 and ϕ_2, respectively. Free to add a huge number to all the labels of ϕ_1, we may assume that $\min\{\phi_1(e) \mid e \in E(G_1)\} >$

$\max\{\phi_2(e) \mid e \in E(G_2)\}$. Then the union of ϕ_1 and ϕ_2 is an (x,y)-better labelling of G. □

Lemma 28. *Let G be a 2-connected ABC-graph with an even number of vertices. If G is $\{C_3, K_{2,3}\}$-free, then G is good.*

Proof: We prove this lemma by induction on the number of vertices (or equivalently the number of A-, B- or C-operations). An even ABC-graph is obtained from K_1 with a sequence of A- and B-operations and exactly one C-operation. Since G is 2-connected, no A-operation can be made after a C-operation. Consider a sequence of operations such that the C-operation is done as late as possible. Let u and v be the vertices on which the C-operation is done and w the introduced vertex.

- Suppose that the C-operation is the ultimate one. Note that $u \neq v$ since G has no multiple edges. Since G is $\{C_3, K_{2,3}\}$-free then u and v are not adjacent and u and v have at most one neighbour in common. Hence by Corollary 27, $G - w$ admits a (u,v)-better edge-labelling ϕ (in $\mathbb{R}$). Setting $\phi(uw) = -\infty$ and $\phi(wv) = +\infty$ we obtain a good edge-labelling of G.

- If the C-operation is the penultimate one, then it is followed by a B-operation on one of the introduced edges, because the C-operation is applied as late as possible and G is C_3-free. These two operations together may be seen as a single one on u and v that introduces the vertices t_1, t_2 and w and the edges ut_1, ut_2, t_1w, t_2w and wv.

  Note that u and v are not adjacent since G is $K_{2,3}$-free. Assume first that u and v have at most one neighbour in common. By Corollary 27, $G - \{t_1, t_2, w\}$ admits a (u,v)-better edge-labelling ϕ. Let M be the maximum value of ϕ. Then setting $\phi(ut_1) = \phi(t_2w) = -\infty$, $\phi(ut_2) = \phi(t_1w) = M+1$ and $\phi(vw) = M+2$, we obtain a good edge-labelling of G.

  Assume now that u and v have at least two common neighbours. Since G is $K_{2,3}$-free, then u and v have exactly two common neighbours x_1 and x_2. By Corollary 23, $G - \{t_1, t_2, w\}$ admits a good edge-labelling ϕ. By Proposition 2, we may assume that ϕ is injective and real-valued. Without loss of generality, we may suppose that $\phi(vx_1) > \phi(vx_2)$. Let us set $\phi(ut_1) = \phi(t_2w) = +\infty$, $\phi(ut_2) = \phi(t_1w) = -\infty$ and $\phi(vw) = \frac{1}{2}(\phi(vx_1)+\phi(vx_2))$. We claim that ϕ is a good edge-labelling of G. Indeed suppose, by way of contradiction, that it is not the case. Then there exist two vertices a and b and two independent increasing (a,b)-paths P_1 and P_2. Since ϕ is a good edge-labelling of $G - \{t_1, t_2, w\}$ one of these two paths, say P_1 must go through w. Moreover since $\phi(t_1w) = -\infty$ and $\phi(t_2w) = +\infty$ and $d(w) = 3$, then either wt_1 (or t_1w) is the first edge of P_1 or t_2w (or wt_2) is the last edge of P_1. Free to consider $-\phi$ instead of ϕ, we may assume that we are in the first case.

  Two cases may occur. Either (a) P_1 starts in t_1 or (b) P_1 starts in w.

(a) In this case, $P_2 = (t_1, u)$ and the third vertex of P_1 is v. Then $Q_1 = P_1 - \{t_1, w\}$ is an increasing (v, u)-path. So by Observation 3.1 and the assumption that $\phi(vx_1) > \phi(vx_2)$, $Q_1 = vx_2u$ (We recall the reader that another increasing (v, u)-path not going through x_2 cannot exist as ϕ is a good edge-labelling of $G - \{t_1, t_2, w\}$). This is a contradiction because $\phi(wv) > \phi(vx_2)$.

(b) In this case, $P_1 = (w, t_1, u)$, because $\phi(ut_1) = +\infty$. Now the first edge of P_2 is wv. Hence $Q_2 = P_2 - w$ is an increasing (v, u)-path and vx_2 is not the first edge of Q_2 since $\phi(wv) > \phi(vx_2)$. Note that by Observation 3.1, vx_2u is increasing because $\phi(vx_1) > \phi(vx_2)$. So, in $G - \{t_1, t_2, w\}$, there are two distinct increasing (v, u)-paths. This contradicts the fact that ϕ is a good edge-labelling of $G - \{t_1, t_2, w\}$.

- If there are exactly two B-operations after the C-operation, and if u and v are not adjacent then by the induction hypothesis and Corollary 21, G has a good edge-labelling. If u and v are adjacent, then uv is a K_2-cut. Let C_1 be the component of $G - \{u, v\}$ containing w (i.e., the set of vertices added with the C-operation and the following B-operations). Let $G_1 = G\langle C_1 \cup \{u, v\}\rangle$ and $G_2 = G\langle V(G) \setminus C_1\rangle$. Note that G_1 is obtained from a triangle by performing two B-operations and thus is the graph G_1 depicted Figure 4 which has a good edge-labelling. Similarly, G_2 is the graph G taken before performing the C-operation has a good edge-labelling. Hence by Lemma 10, G has a good edge-labelling.

- If there are at least three B-operations after the C-operation, then by the induction hypothesis and Corollary 21, G has a good edge-labelling.

□

Lemma 22 and Lemma 28 imply that every 2-connected $\{C_3, K_{2,3}\}$-free ABC-graph is good. Since 2-connected components of an ABC-graph are ABC-graphs, we have the following.

Corollary 29. *Every $\{C_3, K_{2,3}\}$-free ABC-graph is good.*

In turn, this corollary, together with Corollary 12, Theorems 16 and 19, yield the following.

Theorem 30. *Let G be a critical graph. If $G \notin \{C_3, K_{2,3}\}$ then $|E(G)| \geq \frac{3}{2}|V(G)| - \frac{1}{2}$.*

7. Conclusions and further research

We have shown that it is NP-complete to decide if a graph has a good edge-labelling, even for the class of bipartite graphs. It would be nice to find large classes of graphs for which it is polynomial-time decidable. For graphs with treewidth 1, which are the forests, it is the case. But is it also the case for graphs with treewidth at most k?

Problem 31. *Let $k \geq 2$ be a fixed integer. Does there exist a polynomial-time algorithm that decides if a given graph of treewidth at most k has a good edge-labelling?*

We also do not know what is the complexity of the problem when restricted to planar graphs.

Problem 32. *Does there exist a polynomial-time algorithm that decides if a given planar graph has a good edge-labelling?*

We do not even know if there are planar critical graphs distinct from C_3 and $K_{2,3}$.

Problem 33. *Does there exist a $\{C_3, K_{2,3}\}$-free planar graph which is bad?*

If there is no such graphs or only a finite number of them then the answer to Problem 32 will be yes.

Corollary 18 implies that, with the additional condition of girth at least 6, the answer to Problem 33 is no. It would be nice to solve the above problems for planar graphs of smaller girth. In particular, we do not know if there is a planar graph with girth 5 which is bad.

Problem 34. Does every planar graph of girth at least 5 have a good edge-labelling?

Bonsma [4] showed that it is NP-complete to decide if a planar graph of girth at least 5 has a matching-cut. In particular, there are infinitely many planar graphs of girth at least 5 without matching-cut. However, for all such graphs we looked at, we were able to find a good edge-labelling.

The *average degree* of a graph G is $Ad(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.

Theorem 30 implies that for any $c < 3$ there is a finite number of critical graphs with average degree at most c. Actually, we conjecture that the only ones are C_3 and $K_{2,3}$.

Conjecture 35. *Let G be a critical graph. Then $Ad(G) \geq 3$ unless $G \in \{C_3, K_{2,3}\}$.*

More generally for any $c < 4$, we conjecture the following.

Conjecture 36. *For any $c < 4$, there exists a finite list of graphs $\mathcal{L}$ such that if G is a critical graph with $Ad(G) \leq c$ then $G \in \mathcal{L}$.*

The constant 4 in the above conjecture would be tight. Indeed, for all k, the graph $J_{2,2,k}$ defined in Section 3 is critical: it is bad according to Proposition 4. Moreover one can easily show that for any edge e, $H_k \setminus e$ has a good edge-labelling with no (u, v)-increasing path and no (v, u)-increasing (just follow the constraint as in the proof of Proposition 3). Extending this labelling by labelling the two H_2 with $-\infty$ and $+\infty$ such that one of them is an increasing (u, v)-path and the other one an increasing (v, u)-path we obtain a good edge-labelling of $J_{2,2,k} \setminus e$. Furthermore $Ad(J_{2,2,k}) = \frac{8k+8}{2k+4} = 4 - \frac{4}{k+2}$. Last, one can easily see that if $k \neq k'$ then $J_{2,2,k}$ is not a subgraph of $J_{2,2,k'}$.

Theorem 30 says that if a graph has no dense subgraphs then it has a good edge-labelling. On the opposite direction one may wonder what is the minimum density ensuring a graph to be bad. Or equivalently,

Problem 37. What is the maximum number $g(n)$ of edges of a good graph on n vertices?

Clearly we have $g(n) = ex(n, \mathcal{C})$ where $\mathcal{C}$ is the set of critical graphs. As $K_{2,3}$ is critical then $g(n) \leq ex(n, K_{2,3}) = \frac{1}{\sqrt{2}} n^{3/2} + O(n^{4/3})$ by a result of Füredi [8].

The hypercubes show that g is super-linear. Indeed the hypercube H_k is obtained from two disjoints copies of H_{k-1} by adding a perfect matching between them. Hence an easy induction and Lemma 11 shows that H_k has a good edge-labelling. Since H_k has 2^k vertices and $2^{k-1}k$ edges, $g(2^k) \geq 2^{k-1}k$, so $g(n) \geq \frac{1}{2} n \log n$.

References

[1] Bala, K., Stern, T., 1991. Algorithms for routing in a linear lightwave network. In: IEEE INFOCOM'91. Proceedings. Tenth Annual Joint Conference of the IEEE Computer and Communications Societies. Networking in the 90s. pp. 1–9.

[2] Bermond, J.-C., Cosnard, M., Pérennes, S., May 2009. Directed acyclic graphs with unique path property. Tech. Rep. RR-6932, INRIA.

[3] Bonsma, P., 2006. Sparse cuts, matching-cuts and leafy trees in graphs. Ph.D. thesis, University of Twente, Enschede, the Netherlands.
URL `http://purl.org/utwente/57117`

[4] Bonsma, P., 2009. The complexity of the matching-cut problem for planar graphs and other graph classes. J. Graph Theory 62 (2), 109–126.

[5] Bonsma, P., Farley, A. M., Proskurowski, A., 2011. Extremal graphs having no matching cuts. Journal of Graph Theory, n/a–n/a.
URL `http://dx.doi.org/10.1002/jgt.20576`

[6] Eaton, N., Hull, T., 1999. Defective list colorings of planar graphs. Bull. Inst. Combin. Appl 25, 79–87.

[7] Farley, A., Proskurowski, A., 1984. Extremal graphs with no disconnecting matching. In: Proceedings of the second West Coast conference on combinatorics, graph theory, and computing. pp. 153–165.

[8] Füredi, Z., 1996. New asymptotics for bipartite Turán numbers. J. Combin. Theory Ser. A 75 (1), 141–144.

[9] Mukherjee, B., 1997. Optical communication networks. McGraw-Hill New York.

[10] Ramaswami, R., Sivarajan, K., 1995. Routing and wavelength assignment in all-optical networks. IEEE/ACM Transactions on Networking (TON) 3 (5), 489–500.

[11] Schaefer, T. J., 1978. The complexity of satisfiability problems. In: STOC '78: Proceedings of the tenth annual ACM symposium on Theory of computing. ACM, New York, NY, USA, pp. 216–226.

[12] Sereni, J.-S., 2006. Private communication.